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Three-Dimensional Free Convection Flow Near a Two-Dimensional Isothermal Surface

W. H. H. BANKS

Mathematics Department, University of Bristol, United Kingdom

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SUMMARY

This paper is concerned with the existence of a three-dimensional solution to the free convection boundary layer equations near a two-dimensional isothermal surface. The numerical solution is presented in detail, and it is noted that the resulting heat transfer at the body surface is over 25% less than in the two-dimensional solution.

1. Introduction

The flow of a fluid arising as a result of free convection has provided the fluid dynamacist with a variety of interesting phenomena which has resulted in a great deal of research activity both theoretical and experimental.

In the theoretical investigation into flows of this type much work has been done by invoking Prandtl's boundary layer concept and it is not surprising that, in one form or another, the flow occasioned by a vertical plate at a higher temperature than the surrounding fluid has been the topic of many previous investigations (see references in Ostrach [1] and Gebhart [2]). Another free convection flow problem which has stimulated considerable interest concerns the flow field near a lower stagnation point. The adjective "lower" implies that the flow at the outer edge of the boundary layer is towards the surface so that the flow can be considered as one of attachment (i.e. the fluid attaching itself to the body surface). By supposing the body surface to be isothermal and assuming a two-dimensional flow field to exist, the problem has been considered, directly or indirectly, by Hermann [3], Prins and Merk [4], Chen [5] and Poots [6], whereas the possibility of an axisymmetric flow field existing has been considered by Chiang *et al.* [7], and Poots [6].

The work of Poots is particularly interesting because he derives the boundary layer equations governing the free convection flow at a general three-dimensional lower stagnation point, and shows that the two-dimensional and axisymmetric flows are just two special cases out of a continuous spectrum. Poots has in fact given exact numerical solutions to the three-dimensional boundary layer equations, when the Prandtl number is 0.72, for a number of blunt body shapes.

The purpose of this note, however, is to suggest that other solutions (referred to as dual solution) exist over the whole range of stagnation points and Prandtl numbers, and, in particular, to present a detailed numerical dual solution for the flow near a geometrical two-dimensional stagnation point with a Prandtl number of 0.72.

2. The Boundary Layer Equations

Poots has shown that the relevant boundary layer equations are*

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2} + g\beta(T - T_\infty)ax\,, \tag{1}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \nu\frac{\partial^2 v}{\partial z^2} + g\beta(T - T_\infty)by\,, \tag{2}$$

* The notation differs trivially from that used by Poots.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\,, \tag{3}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=k\frac{\partial^2 T}{\partial z^2} \tag{4}$$

where (x, y, z) are cartesian co-ordinates with origin 0 at the stagnation point and such that z measures distance normal to the surface at 0; (u, v, w) are the respective velocities, T is the temperature while T_∞ is the temperature at infinity, g is the acceleration due to gravity, β the coefficient of cubical expansion, k is the thermal diffusivity and v the kinematic viscosity. The two remaining parameters a and b are functions of the geometry of the body surface at 0: they are the curvatures of the body measured in the planes $y=0$ and $x=0$ respectively.

The parameters a and b have been assumed non-negative by Poots so that solutions of the resulting equations lead to stagnation points which are nodal points of attachment. However, this restriction fails to recognise the possible existence of saddle points of attachment where either one of a, b is negative. It is hoped to present the results of such an investigation at a later stage, and for the present assume both a and b to be non-negative.

In terms of the Grashof number, defined by $\mathrm{Gr}=\beta g(T_0-T_\infty)/a^3v^2$, where T_0 is the constant wall temperature, we look for a solution by writing*

$$u=va^2x\mathrm{Gr}^{\frac{1}{2}}f'(Z)\,,\quad v=va^2y\alpha\mathrm{Gr}^{\frac{1}{2}}g'(Z)\,,\quad w=-va\mathrm{Gr}^{\frac{1}{4}}(f+\alpha g) \tag{5}$$
$$h(Z)=(T-T_\infty)/(T_0-T_\infty)$$

where $Z=\mathrm{Gr}^{\frac{1}{4}}az$ and $\alpha=(b/a)^{\frac{1}{2}}$. The continuity equation (3) is automatically satisfied by this choice and equations (1), (2) and (4) yield

$$f'''+(f+\alpha g)f''-f'^2+h=0\,, \tag{6}$$
$$g'''+(f+\alpha g)g''-\alpha g'^2+\alpha h=0\,, \tag{7}$$
$$h''+\sigma(f+\alpha g)h'=0 \tag{8}$$

where $\sigma=v/k$ is the Prandtl number and dashes imply differentiation with respect to Z. These equations are to be solved subject to the boundary conditions

$$f(0)=f'(0)=g(0)=g'(0)=0\,,\quad h(0)=1\,, \tag{9}$$
$$f'(Z)\to 0\,,\quad g'(Z)\to 0\,,\quad h(Z)\to 0 \text{ as } Z\to\infty\,.$$

As Poots points out, when $\alpha=0$ we can recover the well-known two-dimensional problem by assuming $g\equiv 0$, while for $\alpha=1$ the assumption that $f=g$ gives rise to the axisymmetric problem. The general problem, where f, g and h are functions of the independent space variable Z and the two parameters σ and α involves a large amount of computing time and Poots took $\sigma=0.72$ and $0\leqq\alpha\leqq 1$. However, it is of interest to note that the latter restriction is not important because, as can easily be verified, the ordinary differential equations (6), (7) and (8) imply that

$$f(\alpha, Z)=\alpha^{-\frac{1}{2}}g(\alpha^{-1}, \alpha^{\frac{1}{2}}Z)\,,\quad g(\alpha, Z)=\alpha^{-\frac{1}{2}}f(\alpha^{-1}, \alpha^{\frac{1}{2}}Z)\,,\quad h(\alpha, Z)=h(\alpha^{-1}, \alpha^{\frac{1}{2}}Z) \tag{10}$$

and so solutions for values of α such that $0\leqq\alpha\leqq 1$ can be used to generate solutions for $1\leqq\alpha\leqq\infty$, simply by applying (10).

Before proceeding it will be convenient to consider the asymptotic behaviour of f, g and h from equations (6), (7) and (8) for large Z.

3. Asymptotic Behaviour

Because of the boundary condition (9) at $Z=\infty$, we write

$$f=\delta_1+\phi\,,\quad g=\delta_2+\gamma\,,\quad h=\chi$$

* There appears to be typographical errors in the analogous expressions quoted by Poots.

where δ_1 and δ_2 are constants that depend on σ and α, and the functions ϕ, γ and χ, which also depend on σ and α, are functions of Z that are assumed small so that products of these terms can be neglected compared with linear terms. Substitution into (6), (7) and (8) yields (on ignoring small terms) the equations

$$\phi''' + \theta\phi'' = -\chi \tag{11}$$
$$\gamma''' + \theta\gamma'' = -\alpha\chi \tag{12}$$
$$\chi'' + \sigma\theta\chi' = 0 \tag{13}$$

where $\theta = \delta_1 + \alpha\delta_2$.

The solution of (11), (12) and (13) for $\sigma \neq 1$ leads to

$$f = \delta_1 - A\lambda e^{-\sigma\theta Z} + B e^{-\theta Z},$$
$$g = \delta_2 - \alpha A\lambda e^{-\sigma\theta Z} + C e^{-\theta Z},$$
$$h = A e^{-\sigma\theta Z},$$

where $\lambda^{-1} = \sigma^2\theta^3(1-\sigma)$, whereas if $\sigma = 1$ then

$$f = \delta_1 - A\theta^{-2} Z e^{-\theta Z} + B e^{-\theta Z},$$
$$g = \delta_2 - \alpha A\theta^{-2} Z e^{-\theta Z} + C e^{-\theta Z},$$
$$h = A e^{-\theta Z}.$$

A, B and C are constants which can be determined from the numerical solution.

Two points should be noted at this juncture. First, the derivation of the asymptotic forms assumes that the constant θ is strictly positive in order that f', g' and h should vanish as $Z \to \infty$; this is clearly equivalent to the requirement that the flow is one of attachment. Secondly, even for $\sigma = 1$, no vorticity decay arguments are necessary to reject certain solutions. In particular, it does not appear possible to choose the solution corresponding to $A = 0$.

Although these asymptotic forms derived here are useful (and clearly necessary incidentally) before attempting to compute a numerical solution, they do contain the, as yet, unknown θ so that it is not possible to choose the end of the range of integration at this stage.

We shall, as indicated previously restrict attention to the case when $\sigma = 0.72$, but there is no obvious reason why the same sort of dual solution should not exist at other finite values of the Prandtl number. Reference to the situation when $\sigma \to \infty$ is made in section 5.

4. The Dual Solution

We show that, at $\alpha = 0$, a solution exists the resulting flow field being of a three-dimensional nature, and as such is quite distinct from the usual two-dimensional form considered by previous authors. Poots assumes that when $\alpha = 0$, $g(Z) \equiv 0$ so that the equations (6) and (8) become

$$f''' + ff'' - f'^2 + h = 0, \tag{14}$$
$$h'' + \sigma f h' = 0, \tag{15}$$

subject to

$$f(0) = f'(0) = h(0) - 1 = 0,$$
$$f'(Z) \to 0, \quad h(Z) \to 0 \text{ as } Z \to \infty.$$

However, following a parallel investigation by Davey and Schofield [8] into the forced convection problem, another solution is possible by first writing $g = G/\alpha$ and (F, H) instead of (f, h); substituting these forms into (6), (7) and (8) and taking the limit as $\alpha \to 0$, we get

$$F''' + (F+G)F'' - F'^2 + H = 0, \tag{16}$$
$$G''' + (F+G)G'' - G'^2 = 0, \tag{17}$$
$$H'' + \sigma(F+G)H' = 0, \tag{18}$$

subject to the same conditions as in (9). It should be noted here that the velocity in the third dimension (i.e. in the Oy-direction) is, by virtue of (5), given by

$$v = \nu a^2 y \mathrm{Gr}^{\frac{1}{4}} G'(Z)$$

and so is a non-zero physical component, provided a non-zero function $G(Z)$ exists.

Now, although $G \equiv 0$ is clearly one solution, it transpires that another non-zero solution of (17) exists. The asymptotic expansions for F and H have leading terms identical to those found in section 3 for f and h provided that we interpret θ as

$$\theta = \Delta_1 + \Delta_2 ,$$

where $\Delta_1 = \lim_{Z \to \infty} F(Z)$ and $\Delta_2 = \lim_{Z \to \infty} G(Z)$. The behaviour of $G(Z)$ is then given by

$$G(Z) = \Delta_2 + C e^{-\theta Z}$$

where C is a constant.

A numerical solution to equations (16), (17) and (18) with $\sigma = 0.72$ has been obtained using the technique known as the shooting method. Briefly, sets of trial values were assumed for $H'(0)$, $F''(0)$, $G''(0)$ and the equations integrated* from $Z=0$ to $Z=6$. The guessed values of $H'(0)$, $F''(0)$, $G''(0)$ were changed until, on integration, the boundary conditions $H(6) = F'(6) = G'(6) = 0$ were satisfied. The range of integration was progressively increased and the above process repeated until the terms $H'(Z)$, $F''(Z)$, $G''(Z)$ in addition to $H(Z)$, $F'(Z)$, $G'(Z)$ were negligibly small. This was achieved at $Z=21$.

The essential properties for the determination of H, F and G are

$$H'(0) = -0.27809 , \quad F''(0) = 0.96171 , \quad G''(0) = -0.28212$$

which results in $\Delta_1 = 2.618$ and $\Delta_2 = -1.618$. These results should be compared with the solution of (14) and (15) as computed by Poots (and checked during this investigation by the present author) who finds that

$$h'(0) = -0.37411 , \quad f''(0) = 0.85604 , \quad g(Z) \equiv 0 .$$

which results in $\Delta_1 = 1.333$. Table 1 shows the variation of F', G' and H with Z, and for comparison also gives the two-dimensional results $f'(Z)$ and $h(Z)$.

An interesting point to notice here is that $-h'(0)$, which is a measure of the heat transfer at the stagnation point, is reduced by just over 25%. Another point of interest concerns the thickness of the boundary layer: this is indicated by the approach to the boundary conditions at $Z = \infty$ and which is reflected in the exponent in the exponential terms of the asymptotic expansion $-\theta = 1.333$ for the two-dimensional solution as compared with 1.000(!) for the present solution, indicating an increase in the boundary layer thickness of some 25%.

The flow at stagnation points is classified according to the behaviour of the skin-friction lines on the body surface, and it is clear that the three-dimensional solution presented here corresponds to a saddle point of attachment. A diagrammatic sketch is given in figure 1.

5. Large Prandtl Number

Although the only dual solution presented in this investigation is for $\alpha = 0$ and $\sigma = 0.72$, there can be little doubt of their existence for other values of α (as in the forced convection problem: Schofield and Davey [9]) and other finite Prandtl numbers.

For very large Prandtl numbers, it is known that the problem can be posed as a singular perturbation one with small parameter σ^{-1} (see [10] or [11]). We proceed by writing

$$f(Z) = \sigma^{-\frac{3}{4}} f_1(\eta) , \quad g(Z) = \sigma^{-\frac{3}{4}} g_1(\eta) , \quad h(Z) = h_1(\eta)$$

where $\eta = \sigma^{\frac{1}{4}} Z$, so that the basic equations (6), (7) and (8) become

* A computer library procedure was used for the integration which had an automatic change of step-length facility for improving the accuracy.

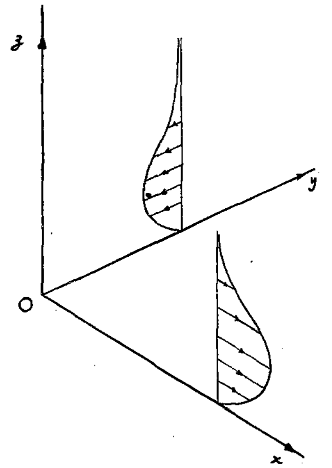

Figure 1. Diagrammatic sketch of the (u, v) velocity field. The Oz velocity component, w, is not shown but is towards the body surface, $z=0$, for all z.

TABLE 1

Z	F'	f'	$-G'$	g'	H	h
0	0	0	0	0	1.0	1.0
0.2	0.1728	0.1518	0.0564	0	0.9444	0.9252
0.4	0.3087	0.2671	0.1125	0	0.8889	0.8506
0.6	0.4119	0.3501	0.1676	0	0.8338	0.7766
0.8	0.4869	0.4053	0.2205	0	0.7793	0.7039
1.0	0.5385	0.4372	0.2701	0	0.7260	0.6333
1.2	0.5710	0.4502	0.3149	0	0.6741	0.5654
1.4	0.5883	0.4483	0.3537	0	0.6240	0.5011
1.6	0.5938	0.4350	0.3855	0	0.5760	0.4409
1.8	0.5903	0.4137	0.4094	0	0.5303	0.3852
2.0	0.5800	0.3868	0.4251	0	0.4870	0.3344
2.2	0.5648	0.3566	0.4327	0	0.4463	0.2886
2.4	0.5459	0.3249	0.4325	0	0.4080	0.2476
2.6	0.5245	0.2929	0.4252	0	0.3722	0.2114
2.8	0.5014	0.2617	0.4117	0	0.3388	0.1797
3.0	0.4772	0.2320	0.3931	0	0.3078	0.1521
3.4	0.4272	0.1787	0.3450	0	0.2524	0.1079
3.8	0.3771	0.1346	0.2895	0	0.2051	0.0757
4.2	0.3286	0.0996	0.2337	0	0.1653	0.0527
4.6	0.2829	0.0727	0.1826	0	0.1320	0.0364
5.0	0.2404	0.0524	0.1387	0	0.1045	0.0251
5.4	0.2019	0.0375	0.1029	0	0.0821	0.0172
5.8	0.1677	0.0266	0.0748	0	0.0640	0.0118
6.2	0.1377	0.0187	0.0535	0	0.0495	0.0081
6.6	0.1120	0.0131	0.0378	0	0.0381	0.0055
7.0	0.0903	0.0092	0.0264	0	0.0292	0.0038
8.0	0.0508	0.0037	0.0104	0	0.0147	0.0014
9.0	0.0275	0.0014	0.0039	0	0.0073	0.0006
10.0	0.0145	0.0006	0.0015	0	0.0036	0.0002
12.0	0.0038	0.0001	0.0002	0	0.0009	0.0000
14.0	0.0009	0.0000	0.0000	0	0.0002	
16.0	0.0002			0	0.0000	
18.0	0.0000					

$$f_1''' + h_1 = \sigma^{-1}\{f_1'^2 - (f_1 + \alpha g_1) f_1''\}\,, \tag{19}$$

$$g_1''' + \alpha h_1 = \sigma^{-1}\{\alpha g_1'^2 - (f_1 + \alpha g_1) g_1''\}\,, \tag{20}$$

$$h_1'' + (f_1 + \alpha g_1) h_1' = 0\,, \tag{21}$$

where dashes now imply differentiation with respect to η. The boundary conditions are the same as in (9) with Z replaced by η.

On taking the limit as $\sigma \to \infty$, we obtain

$$f_1''' + h_1 = 0\,, \quad g_1''' + \alpha h_1 = 0\,, \quad h_1'' + (f_1 + \alpha g_1) h_1' = 0$$

for the first approximation. The boundary conditions to be imposed are

$$f_1(0) = f_1'(0) = g_1(0) = g_1'(0) = h_1(0) - 1 = 0\,,$$
$$f_1''(\eta) \to 0\,, \quad g_1''(\eta) \to 0\,, \quad h_1(\eta) \to 0 \text{ as } \eta \to \infty\,.$$

The boundary conditions $f_1'(\eta) \to 0$, $g_1'(\eta) \to 0$ as $\eta \to \infty$ in the exact equations cannot now be applied, and instead we impose zero shear stress at $\eta = \infty$. The physical reason for this is because, as σ becomes very large the thermal layer becomes much thinner than the momentum layer, so that at the edge of the thermal layer the velocities are still finite.

It follows that

$$g_1 = \alpha f_1\,, \quad h_1 = -f_1'''$$

where f_1 is determined by the equation

$$f_1^{v} + (1 + \alpha^2) f_1 f_1^{iv} = 0\,, \tag{22}$$

subject to

$$f_1(0) = f_1'(0) = 0\,, \quad f_1'''(0) = -1\,,$$
$$f_1''(\eta) \to 0\,, \quad f_1'''(\eta) \to 0 \text{ as } \eta \to \infty\,.$$

Finally, the transformation

$$f_1(\eta) = (1 + \alpha^2)^{-\frac{3}{4}} f_{11}(\xi)\,,$$

where $\xi = (1 + \alpha^2)^{\frac{1}{4}} \eta$, enables equation (21) to be written

$$f_{11}^{v} + f_{11} f_{11}^{iv} = 0\,, \tag{23}$$

subject to

$$f_{11}(0) = f_{11}'(0) = 0\,, \quad f_{11}'''(0) = -1\,,$$
$$f_{11}''(\xi) \to 0\,, \quad f_{11}'''(\xi) \to 0 \text{ as } \xi \to \infty \tag{24}$$

which is completely free of the parameter α.

Equation (23) subject to (24) coincides precisely with the equation arising in a two-dimensional study by Le Fevre [10]: he gives

$$f_{11}''(0) = 1.085125\,, \quad f_{11}^{iv}(0) = 0.540235\,.$$

The reduction of the full three-dimensional equations to two-dimensional form for very large Prandtl numbers is a very interesting result, although since no dual solution of (23) subject to (24) is known, it does then follow that there are no dual solutions for any α in this case. However, this is only strictly true in the limit as $\sigma \to \infty$, resulting in the inertia terms of equations (19) and (20) being ignored, and will presumably be untrue for large but finite Prandtl numbers.

6. Conclusion

The existence of a three-dimensional solution at a two-dimensional stagnation point has been exhibited by way of a numerical solution for a Prandtl number $\sigma = 0.72$. Also investigated is the behaviour at a general three-dimensional stagnation point for infinitely large Prandtl numbers. It is shown that in the limit as $\sigma \to \infty$ the problem can be reduced to the two-dimensional case.

The physical meaning of such dual solutions is still not understood, although Davey and Schofield [8] do suggest that they may be interpreted as a finite disturbance to the usual solution, and, as such, may be related to the general instability of such flows. It was with the possibility of experimental verification in mind and the hope that it may shed light on bifurcation phenomena in general, that work on this project was started.

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